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Limit Cycles in Switched Single-Server Flow Networks

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Abstract—The paper addresses the problem of qualitative analysis for a class of hybrid dynamical systems. This class consists of so-called switched flow networks which are used to model various communication, computer, and flexible manufacturing systems. We prove that any hybrid dynamical system from this class has a finite number of asymptotically stable limit cycles and any trajectory of the system converges to one of these cycles.

Keywords—Hybrid dynamical systems, Stable limit cycles, Logic-differential equations, Switched flow networks.

1. INTRODUCTION

Hybrid dynamical systems (HDS) have attracted considerable attention in recent years. In general, hybrid dynamical systems are those that combine continuous and discrete dynamics and involve both continuous and discrete state variables. From an engineering viewpoint, a hybrid system is a network of digital and analog devices or a digital device that interacts with a continuous environment. These systems typically contain variables that take values from a continuous set (usually, the set of real numbers) and also variables that take values from a discrete set (e.g., the set of symbols $\{q_1, q_2, \ldots, q_n\}$). A simple example is a home climate-control system. Due to its on-off nature, the thermostat is modeled as a discrete-event system, whereas the furnace or air-conditioner are modeled as continuous-time systems. Some other examples of hybrid systems include automotive power train systems, computer disk drives, robotic systems, automotive engine management, high-level flexible manufacturing systems, intelligent vehicle/highway systems, sea/air traffic management, modern spacecraft control systems, job scheduling, interconnected power systems, chemical processes. In fact, many problems facing engineers and scientists as they seek to use computers to control complex physical systems naturally fit into the HDS framework. The study of hybrid dynamical systems represents a difficult and exciting challenge in control engineering. This field is referred to as "the control theory of tomorrow" by SIAM News [1].

This paper studies a class of hybrid dynamical systems, which are called switched flow networks. Special classes of such networks were introduced in [2] to model flexible manufacturing assem-

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bly/disassembly systems. Furthermore, these networks are useful to model various computer and communication systems, especially those with time-sharing schemes.

It is known that even very simple flow networks of order 2 can exhibit a chaotic irregular unpredictable behavior [3,4]. Such a behavior is unacceptable for many real world engineering systems. A typical synthesis problem (e.g., see [4–6]) is to find a feedback switching policy that guarantees a regular predictable behavior of the network. This problem involves qualitative analysis of the corresponding closed-loop system. However, the papers [3–6] considered only two-dimensional systems.

In this paper, we consider quite general networks of arbitrary dimensions. More precisely, we assume that the network consists of an arbitrary number of buffers (nodes) connected with links (edges). The content of the buffers (called work) arrives from outside the system at certain buffers. The network is processed by a single server. It removes work from a selected buffer and delivers it along the edges departing from this buffer. The location of the server is a discrete control variable and is determined by a feedback policy. We study a quite natural switching policy combining the clear-the-largest-buffer-level strategy [2] and the cyclic policy [6,7]. The case of the cyclic switching policy for some simple switched flow networks was studied in [7–9]. The main result of the current paper shows that the corresponding closed-loop system exhibits a globally periodic behavior. More precisely, this hybrid system has a finite number of locally asymptotically stable limit cycles and any trajectory of the system converges to one of these limit cycles.

To obtain criteria of existence of self-excited oscillations or limit cycles is a very old and challenging problem of the classic qualitative theory of differential equations whose origins may be traced back to the work of Poincaré and Lyapunov; e.g., see [6]. Few constructive results are known for nonlinear systems of order higher than 2. It is even harder to study stability of limit cycles. The results presented in this paper show that constructive criteria of existence and stability of limit cycles can be established for quite general switched flow networks. This appears to be surprising and gives us a hope that it is possible to develop a qualitative theory of some classes of hybrid dynamical systems, which will be even more constructive than the classic qualitative theory of differential equations.

The proofs of the main results of the current paper are based on the theory developed in [11]. The proofs are available upon request and will be published elsewhere.

2. SINGLE SERVER FLOW NETWORKS

Consider an oriented graph $\mathfrak{G}$ with the set of the nodes

$$\hat{G} := \{g_1, \dots, g_L, g_{L+1} = \infty\}.$$

The edge departing from g_i and arriving at g_j is denoted by (g_i, g_j). (There is no more than one such edge.) The special node ∞ is interpreted as the exterior of the system. Correspondingly, any edge of the form (∞, g_i) or (g_i, ∞) (where $i = 1, \dots, L$) is regarded as coming from outside or, respectively, going outside the system.

ASSUMPTION 1. *The set of the nodes $\hat{G}$ can be partitioned into a finite number of nonempty subsets*

$$\hat{G} = S_0 \cup S_1 \cup \cdots \cup S_M,$$

so that the following requirements are satisfied.

- *The set S_0 consists of the only element $S_0 = \{\infty\}$.*
- *Any edge departing from a node from S_i arrives at a node from S_{i+1}. Here $i = 0, \dots, M$ and $S_{M+1} := S_0 = \{\infty\}$.*
- *For any node g_j $(j = 1, \dots, L)$, there exists an edge arriving at g_j and an edge departing from g_j.*

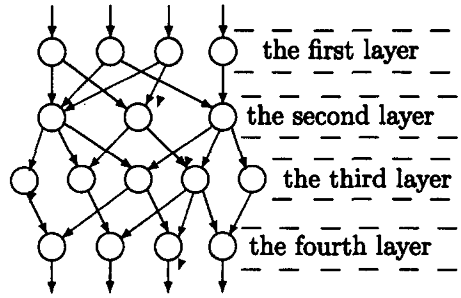

Figure 1. A flow network.

DEFINITION 1. *The set* S_i *(where* $i = 1, \dots, M$*) is called the* i^{th} *layer of the graph.*

Associated with each node

$$g \in G := \{g_1, \dots, g_L\}$$

is a buffer (or tank). Its content is called "work" and interpreted as fluid. The work arrives to the system continuously along the edges of the form (∞, g) at a constant rate $\rho_g > 0$. (Here $g \in S_1$.) There also is a server (or machine), which serves buffers. At any time, the server is processing one of the buffers. The server removes the work from a selected buffer g at a constant rate $\rho > 0$ and delivers it along the edges departing from g. The constant ρ does not depend on the buffer. The distribution of the work flow among the edges is in a given proportion. In other words, the server sends work along the edge (g, g') at a constant rate $\rho(g, g') > 0$ and

$$\sum_{g' \in G(g)} \rho(g, g') = \rho, \qquad \forall g \in G. \tag{1}$$

Here $G(g)$ is the set of the nodes $g' \in \hat{G}$ such that (g, g') is an edge. (Note that $G(g) \subset S_{i+1}$ whenever $g \in S_i$.) The location of the server is a control variable, which is chosen in accordance with a prescribed feedback control policy. We assume that the server switches between buffers instantaneously.

Let x_g be the content of the buffer g.

REMARK 1. Assume that the network contains at least two buffers. Then, for any server switching policy, the following two statements hold.

(i) If

$$\sum_{g \in S_1} \rho_g > M^{-1} \rho, \tag{2}$$

then the total amount of work in the system

$$x(t) := \sum_{g \in G} x_g(t)$$

tends to ∞ as $t \to \infty$.

(ii) If

$$\sum_{g \in S_1} \rho_g < M^{-1} \rho, \tag{3}$$

then there exists a finite time $t_* \geq 0$ such that the server switches infinitely many times over the time interval $[0, t_*]$.

Hence, the only sensible case is that with

$$\sum_{g \in S_1} \rho_g = M^{-1} \rho. \tag{4}$$

This case corresponds to the class of closed switched flow systems; e.g., see [3]. It is easy to see that if (4) holds, then the scaled total amount of work in the system,

$$\sigma(t) := \sum_{i=1}^{M} \frac{M-i+1}{M} \sum_{g\in S_i} x_g, \tag{5}$$

is constant.

3. A SWITCHING CONTROL POLICY

In this paper, we consider the following switching policy.

SP1 The server passes the layers consecutively. At first, it serves buffers from the first layer, then runs over all the buffers from the second one and so on. When all the buffers from the last layer have been served, it returns to the first layer.

SP2 The first layer is served on the basis of the clear-the-largest-buffer-level strategy [2]. This means that the server switches as soon as the current buffer is emptied to the buffer $g \in S_1$ with the largest scaled content $\zeta_g := \rho_g^{-1} x_g$. Likewise, the first layer service session begins with the buffer $g \in S_1$ having the largest value of ζ_g.

SP3 The first layer service session ends when the server has changed buffers $k-1$ times and emptied the last buffer $g \in S_1$ to which it has been switched. (Here k is the number of the buffers in S_1.)

SP4 After this, the server switches to a given buffer $g_{\text{initial}}^{(2)} \in S_2$ from the second layer and is governed afterwards as follows. As soon as a current buffer g is emptied, the server switches to the buffer

$$\eta(g) \in S := S_2 \cup \cdots \cup S_M.$$

In other words, the sequence of the further buffer changes is as follows:

$$g_0 := g_{\text{initial}}^{(2)} \mapsto g_1 := \eta(g_0) \mapsto g_2 := \eta(g_1) \mapsto \cdots \mapsto g_{\hat{L}-1} := \eta\left(g_{\hat{L}-2}\right). \tag{6}$$

Here $\hat{L}$ is the number of the buffers in S and $\eta(\cdot) : S \to S$ is a given permutation of S satisfying the following assumption.

ASSUMPTION 2. *Sequence (6) first runs over all the buffers in S_2, then ranges over all the buffers in S_3, and so on concluding with running over S_M.*

SP5 As soon as the server empties the last buffer $g_{\hat{L}-1}$ from sequence (6), it starts a new session of serving the first layer and so on.

In some cases, the server may be switched to an empty buffer in accordance with the above policy. Thus, the server can make several instantaneous buffer changes until it reaches a nonempty buffer.

The state of the system is described by a pair (x, q) consisting of the "continuous" component x and the "discrete" component q. Here $x = \{x_g\}_{g\in G}$ and

$$q \in Q := S_2 \cup \cdots \cup S_M \cup \{(g,i)\}_{g\in S_1, i=1,\dots,k}.$$

If the server is in the buffer g at the time t, then $q(t) = g \in S_2 \cup \cdots \cup S_M$. If the server is processing the buffer $g \in S_1$ at the time t and this buffer is the i^{th} in the current session of serving the first layer, then $q(t) = (g, i)$.

The evolution of the system is described by the following logic-differential equations.

If

$$\left\{\begin{array}{l} q = g \in S_2 \cup \cdots \cup S_M \\ \text{or} \\ q = (g,i), \in S_1,\ i = 1,\dots,k \end{array}\right\}, \quad \text{then} \left\{\begin{array}{ll} \dot{x}_{g'} = \rho_{g'}, & \text{whenever } g' \in S_1 \text{ and } g' \neq g, \\ \dot{x}_g = \left\{\begin{array}{ll} -\rho, & \text{if } g \bar{\in} S_1, \\ \rho_g - \rho, & \text{if } g \in S_1, \end{array}\right. & \\ \dot{x}_{g'} = \rho(g, g'), & \text{if } g' \in G(g), \\ \dot{x}_{g'} = 0, & \text{otherwise.} \end{array}\right.$$

If

$$q(t) = \left\{ \begin{array}{l} (g,i),\ g \in S_1,\ i = 1,\dots,k \\ \text{or} \\ g \in S_2 \cup \cdots \cup S_M \end{array} \right\} \text{ and } x_g(t) = 0, \qquad \text{then}$$

$$q(t+0) := \begin{cases} \left. \begin{array}{c} (g', i+1) \text{ if } i < k, \text{ where } g' \in S_1 \\ \text{is such that } \zeta_{g'}(t) \geq \zeta_{g''}(t)\ \forall\, g'' \in S_1 \\ g^{(2)}_{\text{initial}} \text{ if } i = k \end{array} \right\}, & \text{whenever } \begin{array}{c} q(t) = (g,i), \\ g \in S_1,\ i = 1,\dots,k, \end{array} \\ \left. \begin{array}{c} \eta(g) \text{ if } g \neq g_{\widehat{L}-1} \text{ (see (6))} \\ (g', 1) \text{ if } g = g_{\widehat{L}-1}, \text{ where } g' \in S_1 \\ \text{is such that } \zeta_{g'}(t) \geq \zeta_{g''}(t)\ \forall\, g'' \in S_1 \end{array} \right\}, & \text{whenever } q(t) = g \in S_2 \cup \cdots \cup S_M. \end{cases}$$

Any pair of functions $[x(\cdot), q(\cdot)]$ with absolute continuous $x(\cdot)$ and piecewise constant and left-continuous $q(\cdot)$ that satisfy the above equations is called a *trajectory.* A given initial condition may give rise to several trajectories, since the buffer $g' \in S_1$ such that $\zeta_{g'} \geq \zeta_{g''}\ \forall\, g'' \in S_1$ is not determined uniquely in certain cases.

We assume that (4) holds and consider the trajectories with $\sigma(0) = 1$, where the quantity σ is given by (5). The system is studied in the invariant domain

$$K := \{(x,q) : q \in Q,\ x_g \geq 0\ \forall\, g,\ \sigma = 1\}. \tag{7}$$

4. ASYMPTOTIC BEHAVIOR OF THE SYSTEM

For $x = \{x_g\}_{g \in G}\, (x_g \in \mathbf{R})$, let $\|x\| := \sum_{g \in G} |x_g|$. The symbol **mes** denotes the Lebesque measure.

DEFINITION 2. *Let $\mathbf{t}_p = [x_p(\cdot), q_p(\cdot)]$ be a periodic trajectory with a period $T > 0$. A trajectory $[x(\cdot), q(\cdot)], t \in [0, \infty)$ is said to converge to $\mathbf{t}_p$ as $t \to \infty$, if there exists a sequence $\{\tau_i\} \subset (0, +\infty)$ such that $\tau_{i+1} - \tau_i \to T$ as $i \to \infty$ and, for any $\lambda > 0$,*

$$\begin{array}{l} \max\{\|x(t+\tau_i) - x_p(t)\| : t \in [0,\lambda]\} \to 0, \\ \textbf{mes}\ \{t \in [0,\lambda] : q(t+\tau_i) \neq q_p(t)\} \to 0, \end{array} \qquad \textit{as } i \to \infty.$$

If this property holds for some period T of $\mathbf{t}_p$, then it is obviously true for any of them. Furthermore, if a trajectory $\mathbf{t}$ converges to $\mathbf{t}_p$ as $t \to \infty$, then it clearly converges to any trajectory that is a shift $\mathbf{t}_p^{(\tau)}(t) := \mathbf{t}_p(t+\tau)$ ($\tau = \text{const} > 0$) of $\mathbf{t}_p$ in time.

DEFINITION 3. *A periodic trajectory $\mathbf{t}_p = [x_p(\cdot), q_p(\cdot)] \in K$ is said to be locally asymptotically stable in K if there exists $\varepsilon > 0$ such that for any $\theta \geq 0$, any trajectory $\mathbf{t} = [x(\cdot), q(\cdot)] \in K$ with $q(0) = q_p(\theta)$ and $\|x(0) - x_p(\theta)\| < \varepsilon$ converges to $\mathbf{t}_p$ as $t \to \infty$.*

Let a periodic trajectory $\mathbf{t}_p = [x_p(t), q_p(t)] \in K$ be locally asymptotically stable. Then, for any $t_0 \geq 0$, the trajectory $[x_p(t-t_0), q_p(t-t_0)] \in K$ is also locally asymptotically stable.

DEFINITION 4. *Let $\mathbf{t}_p = [x_p(t), q_p(t)] \in K$ be a periodic trajectory. The class of periodic trajectories $\mathfrak{LC} := \{[x_p(t-t_0), q_p(t-t_0)]\}$ is called a limit cycle.*

DEFINITION 5. *A trajectory $\mathbf{t}$ is said to converge to a limit cycle $\mathfrak{LC}$ if it converges to any periodic trajectory from $\mathfrak{LC}$. A limit cycle from K is said to be locally asymptotically stable in K if any periodic trajectory from this limit cycle is locally asymptotically stable.*

Now we are in a position to present the main result of the paper.

THEOREM 1. *Assume that the system contains at least two layers. Furthermore, suppose that the requirements (1),(4) and Assumptions 1 and 2 are satisfied. Let k be the number of the buffers in the first layer of the system. Consider the closed-loop system with the control policy SP1–SP5. Then the following two statements hold.*

(i) *There exist $k! := 1 \times 2 \times \cdots \times k$ locally asymptotically stable in K limit cycles.*
(ii) *Any trajectory of the system from K converges to one of these limit cycles.*

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